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The many-body problem for q-oscillators

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Abstract. The correct Hamiltonian for the N-body problem of free q-oscillators is found, which promotes the symmetries of the standard ($q=1$) oscillator systems to q-symmetries. The spectrum of the system is found to be rich, exhibiting interactions between the levels of the individual oscillators.

1. Introduction

In the operator formulation of quantum field theory, symmetries are realized through the Jordan–Wigner construction using free Fermi or Bose annihilation and creation operators [1]. On the other hand new types of symmetries have been shown to appear in two-dimensional integrable statistical systems, the quantum or q-symmetries, which are one-parameter deformations of the usual Lie algebras (Lie groups) with generalized rules for the tensor product of their representations [2–8]. It was natural, then, to extend the Jordan–Wigner construction by inventing oscillators with appropriately deformed commutation relations [9–10]. Indeed, it has been shown that for most of the q-algebras, or q-superalgebras [11–18] the construction can be carried through successfully. Although the quantum group structure itself has its origin in two-dimensional integrable systems with non-trivial dynamics, the inverse problem of constructing Hamiltonians, for systems with finite or infinite number of degrees of freedom, with a given quantum group symmetry does not yet appear to have been studied systematically.

In this work we show that the construction of the two-dimensional q-oscillator Hamiltonian is not so straightforward if one wants to promote the SU(2) symmetry, of the $q=1$ case, to $SU_q(2)$. Solving this problem one learns how to write down Hamiltonians of many q-oscillators which exhibit, as symmetries, the q-deformations of the standard case. Since the algebra of the single q-oscillator turns out to be the building block for these constructions it is profitable to obtain 'coordinate' realizations of the Fock space of the single q-oscillator. Although realizations exist in the literature, as far as we know the q-deformation of the Bargmann–Fock holomorphic realization of quantum mechanics has not been constructed before. As we shall see this is a straightforward construction with many possible applications.

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1.1. The *q*-oscillator algebra

To set down our convention the q-Heisenberg–Weyl algebra is defined as [9, 10]:

$$aa^+ - qa^+a = q^{-N} \tag{1.1a}$$

where a, a^+ the annihilation and creation operators of the q-oscillator, while the number operator, N, is defined to be

$$a^+a = \frac{q^N - q^{-N}}{q - q^{-1}} \equiv [N]. \tag{1.1b}$$

Here, q is any complex number ($q \neq -1$). In a unitary representation where a, a^+ are Hermitian conjugate, N is an Hermitian operator only if q is real or a phase $q = e^{i\alpha}$ ($\alpha \neq \pi$). If q is a positive real number, positivity of a^+a implies positivity of N. If q is a phase one has to be careful in defining the representation space. The Fock space F_1 is constructed, assuming the existence of a 'vacuum' state, $|0\rangle$, which is annihilated by a and, on this vacuum, 'excited' states are constructed:

$$a|0\rangle = 0 \qquad |n\rangle = \frac{a^{+n}}{\sqrt{[n]!}}|0\rangle \qquad n = 0, 1, 2, \ldots \tag{1.2a}$$

$$[n]! = [n][n-1]\ldots[1] \qquad [x] \equiv \frac{q^x - q^{-x}}{q - q^{-1}}. \tag{1.2b}$$

Then the matrix elements of a, a^+, N are

$$a|n\rangle = \sqrt{[n]}|n-1\rangle \tag{1.3a}$$

$$a^+|n\rangle = \sqrt{[n+1]}|n+1\rangle \tag{1.3b}$$

$$N|n\rangle = n|n\rangle \qquad n = 0, 1, \ldots. \tag{1.3c}$$

In this representation the following relations are true:

$$aa^+ - q^{-1}a^+a = q^N \tag{1.4a}$$

$$q^{\lambda N}a^+q^{-\lambda N} = q^\lambda a^+ \qquad q^{\lambda N}aq^{-\lambda N} = q^{-\lambda}a \tag{1.4b}$$

and the algebra (1.1a) and (1.1b) is equivalent on F_1, with the following 'superalgebra',

$$[N, a] = -a \tag{1.5a}$$

$$[N, a^+] = a^+ \tag{1.5b}$$

$$\{a, a^+\} = \frac{\sinh \gamma(N + \frac{1}{2})}{2\sinh(\gamma/2)} \qquad q \equiv e^\gamma. \tag{1.5c}$$

We must notice that if $\gamma = 2\pi i/k$, $k = 3, 4, \ldots$, a root of unity, there are only κ-states $|0\rangle, |1\rangle, \ldots, |\kappa - 1\rangle$ and the algebra (1.1a), (1.1b) is supplemented by the relations [17]

$$a^\kappa = 0 \qquad (a^+)^\kappa = \beta I \qquad \beta \in \mathbb{C}. \tag{1.6}$$

The q-deformation of the Bargmann–Fock representation is realized by going over the space of analytic functions of one complex variable $z \in \mathbb{C}$, where the operator a, a^+, N, are defined as [8, 19]

$$a = D_z \qquad a^+ = z \qquad N = z\partial_z \tag{1.7a}$$

$$(D_z f)(z) \equiv \frac{f(qz) - f(q^{-1}z)}{z(q - q^{-1})} \qquad \forall z \in \mathbb{C}. \tag{1.7b}$$

In the space of analytic function $f(z)$, $\mathcal{F}$, there is an inner product which makes z and D_z Hermitian conjugates:

$$(f, g) = \overline{f(D_z)}g(z)|_{z=0}. \tag{1.8}$$

The set of functions which represent the states, $|n\rangle$,

$$\langle z|n\rangle \equiv u_n(\bar{z}) = \frac{\bar{z}^n}{\sqrt{[n]!}} \qquad n = 0, 1, 2, \ldots \tag{1.9}$$

constitutes an orthonormal basis with respect to the inner product (8). The 'δ' function which expresses the completeness is

$$\delta(\zeta, \bar{z}) = \sum_{n=0}^{\infty} u_n(\zeta)u_n(\bar{z}) = e_q(\zeta\bar{z}) \tag{1.10}$$

where the q-exponential (an eigenfunction of D_z) is defined as,

$$e_q(z) \equiv \sum_{n=0}^{\infty} \frac{z^n}{[n]!}. \tag{1.11}$$

In the standard ($q = 1$) Bargmann–Fock representation [20] there is a measure $\mathrm{d}\mu(z)$ which realizes the exponential function as a 'δ' function,

$$\int \mathrm{d}\mu(z)\, \mathrm{e}^{\bar{\zeta}z} f(\bar{z}) = f(\bar{\zeta}) \tag{1.12a}$$

this is easily seen to be

$$\mathrm{d}\mu(z) = \mathrm{d}z\, \mathrm{d}\bar{z}\, \mathrm{e}^{-\bar{z}z} \tag{1.12b}$$

(the factor e^{-zz} is the inverse of '$\delta(0)$'). If $q \neq 1$ it is possible to define a q-deformation of the measure $\mathrm{d}\mu(z)$:

$$\mathrm{d}\mu_q(z) = \mathrm{d}_q|z|^2 \frac{\mathrm{d}\varphi}{2\pi} e_q(-\bar{z}z) \tag{1.13}$$

where the q-integration over $\mathrm{d}_q|z|$ is defined as [21]

$$\int_a^b \mathrm{d}_q x f(x) = (q^{-1} - q) \sum_{k=0}^{\infty} q^{2\kappa+1}(bf(q^{2\kappa+1}b) - af(q^{2\kappa+1}a)). \tag{1.14}$$

Indeed, it has been shown recently that the functions $u_n(z)$, $n = 0, 1, 2, \ldots$ from a complete orthonormal system with respect to the measure (1.13), where the radial q-integration is between $0 \leqslant |z| \leqslant X_0$, and $-X_0 < 0$ is the largest zero of the function $e_q(X)$ [21].

This implies that the inner product

$$(f, g) = \int \mathrm{d}\mu_q(z) e_q(-\bar{z}z)\bar{f}(z)g(z) \tag{1.15}$$

is identical with that previously defined in (1.8).

The transition functions $u_n(z)$ are used to define the q-coherent states [21]

$$|z\rangle \equiv N_z \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{[n]!}}|n\rangle = N_z e_q(za^+)|0\rangle \tag{1.16}$$

where

$$N_z = [e_q(\bar{z}z)]^{-1/2} \tag{1.17}$$

and the states $|z\rangle$ satisfy the relation

$$a|z\rangle = z|z\rangle \tag{1.18}$$

as can be easily checked. These states form an over-complete system [21] with respect to the measure $d_q\mu(z)$:

$$\int d\mu_q(z)|z\rangle\langle z| = I \tag{1.19a}$$

$$\langle\zeta|z\rangle = N_\zeta N_z e_q(\bar{\zeta}z) \tag{1.19b}$$

where

$$N_z = (e_q(\bar{z}z))^{-1/2} \tag{1.19c}$$

are the normalization factors.

To every operator A in the Bargmann–Fock space we may associate its symbol $A(\bar{\zeta}, z)$:

$$A(\bar{\zeta}, z) = \langle\zeta|A|z\rangle. \tag{1.20}$$

The symbol has the following interesting properties, which are useful in the holomorphic path integral quantization [1, 20]. If $|f\rangle$ is a state

$$|f\rangle = \int d\mu_q(z) f(\bar{z})|z\rangle \tag{1.21}$$

then

$$A|f\rangle \to (Af)(\bar{\zeta}) = \int d\mu_q(z) A(\bar{\zeta}, z) f(\bar{z}) \tag{1.22a}$$

$$\langle\zeta|A|f\rangle = \int d\mu_q(z) \langle\zeta|A|z\rangle\langle z|f\rangle \tag{1.22b}$$

and

$$(A_1A_2)(\bar{\zeta}, z) = \int d\mu_q(\xi) A_1(\bar{\zeta}, \xi) A_2(\bar{\xi}, z). \tag{1.23}$$

Before leaving the one-oscillator system, we mention that it is possible to realize 'holomorphically' the q-algebra (1.1a), (1.1b) in a finite-dimensional space when q is the κth primitive root of unity:

$$q = \omega = e^{2\pi i/k} \qquad \kappa = 3, 4, \ldots \tag{1.24}$$

In this case, introducing the space of functions on a discretized circle, with points only at the positions of the κ-roots of unity,

$$S_\kappa = \{1, \omega, \ldots, \omega^{\kappa-1}\} \tag{1.25}$$

we define the action of the exponentials of the 'position' and 'momentum' operators [22]

$$(hf)(\omega^n) = f(\omega^{n+1}) \tag{1.26a}$$

$$(gf)(\omega^n) = \omega^n f(\omega^n) \qquad n = 0, 1, \ldots, \kappa - 1. \tag{1.26b}$$

In the base of functions $f_n(\omega^m)=\delta_{nm}$, $n, m=0,1,\ldots,\kappa-1$ the operators h, g (P, Q)

$$h=e^{(2\pi i/k)P} \qquad g=e^{iQ} \tag{1.27}$$

have the matrix representation

$$h=\begin{bmatrix} 0 & 1 & \ldots & 0 \\ & & \ddots & \\ & & & 1 \\ 1 & & & 0 \end{bmatrix} \qquad g=\begin{bmatrix} 1 & & & \\ & \omega & & \\ & & \ddots & \\ & & & \omega^{\kappa-1} \end{bmatrix} \tag{1.28}$$

and they satisfy the following properties:

$$h^k=g^k=I \qquad hh^+=gg^+=I \tag{1.29a}$$

$$hg=\omega gh \qquad UhU^{-1}=g \qquad U_{ij}=\frac{\omega^{ij}}{\sqrt{N}} \qquad i,j=0,\ldots,\kappa-1. \tag{1.29b}$$

The operators a, a^+ in the finite-dimensional 'q-holomorphic' representation are defined to be

$$a=g^{-1}\frac{h-h^{-1}}{\omega-\omega^{-1}} \qquad a^+=g \tag{1.30}$$

where P is the 'angular momentum' operator (1.27),

$$a=g^{-1}[P]. \tag{1.31}$$

Then using the first relation of (1.29b) we find

$$aa^+-\omega a^+a=\omega^{-P} \qquad aa^+-\omega^{-1}a^+a=\omega^P \tag{1.32}$$

so the number operator is

$$N=P. \tag{1.33}$$

The eigenstates of the operator P are [22], $f_m\in\mathscr{F}(S_\kappa)$,

$$f_m(\omega^n)=c_m\omega^n \qquad Pf_m=mf_m \qquad m=0,\ldots,\kappa-1 \tag{1.34}$$

the vacuum f_0,

$$af_0=g^{-1}[P]f_0=0 \tag{1.35a}$$

and the rest f_m, $m=1,2,\ldots,\kappa-1$ turns out to be the 'excited' states:

$$f_m=\frac{(a^+)^m}{\sqrt{[m]!}}f_0 \qquad c_m=\frac{1}{\sqrt{[m]!}} \qquad m=0,1,\ldots,\kappa-1. \tag{1.36}$$

It is easy to check now that

$$a^+f_m=\sqrt{[m+1]}f_{m+1} \qquad m=0,1,\ldots,\kappa-2 \tag{1.37a}$$

$$af_m=\sqrt{[m]}f_{m-1} \qquad m=1,2,\ldots,\kappa-1 \tag{1.37b}$$

as also (see (1.29a))

$$a^+f_{\kappa-1}=\frac{1}{\sqrt{[\kappa+1]!}}f_0 \tag{1.38a}$$

$$af_0=0. \tag{1.38b}$$

The q-harmonic oscillator Hamiltonian problem [10, 11], has been studied for $q^\kappa=1$, in [23], where its relation with the angular motion of the two-anyon system has been found.

The above realization (1.30) leads to the Hamiltonian for the q-oscillator:

$$H=\frac{\sin(2\pi/\kappa)(P+\frac{1}{2})}{2\sin(\pi/\kappa)}=-\frac{\cos(2\pi/\kappa)L_\varphi}{2\sin(\pi/\kappa)}. \tag{1.39}$$

where L_φ is the angular momentum operator for the two-anyon system [23],

$$L_\varphi=P+\frac{\kappa+2}{4}. \tag{1.40}$$

If q is a phase $q=\mathrm{e}^{\mathrm{i}\gamma}$ then it is the limit of an appropriate sequence of roots of unity:

$$q=\mathrm{e}^{\mathrm{i}\gamma}=\lim_{n\to\infty}\mathrm{e}^{\mathrm{i}\gamma_n} \qquad \gamma_n\in Q, n=1,2,\ldots. \tag{1.41}$$

Then the matrices g and h introduced above act on infinite component vectors ($\kappa\to\infty$), which can be thought as the Fourier coefficients of functions on the unit circle $S_1=\{z\in\mathbb{C}, |z|=1\}$ [22],

$$h=\mathrm{e}^{\gamma\partial\theta} \qquad g=\mathrm{e}^{\mathrm{i}\theta} \qquad q=\mathrm{e}^{\mathrm{i}\gamma} \tag{1.42}$$

and

$$a=\mathrm{e}^{-\mathrm{i}\theta}\frac{\mathrm{e}^{\gamma\partial\theta}-\mathrm{e}^{-\gamma\partial\theta}}{\mathrm{e}^{\mathrm{i}\gamma}-\mathrm{e}^{-\mathrm{i}\gamma}} \qquad a^+=\mathrm{e}^{\mathrm{i}\theta} \tag{1.43a}$$

$$N=-\mathrm{i}\partial_\theta. \tag{1.43b}$$

The representation (1.42), (1.43a) and (1.43b) is just the restriction of the q-Bargmann–Fock representation on the unit circle.

Finally we would like to mention that the above realizations of the q-oscillator algebra may be used to study realizations of the q-Virasoro or $q-W$ algebras [14, 24–28]. For the q-Virasoro algebra (centreless), the expressions for the L_n generators [14, 17],

$$L_n=(a^+)^{n+1}a \tag{1.44}$$

lead to the algebra

$$[L_n, L_m]_{q^{m-n}}=[n-m]q^N L_{n+m} \qquad n, m\in\mathbb{Z} \tag{1.45a}$$

$$[A, B]_\lambda\equiv AB-\lambda BA. \tag{1.45b}$$

where N is the number operator for the q-oscillator.

The q-holomorphic realization of (1.44), (1.45a) and (1.45b) is [24]

$$L_n=z^{n+1}D_z \tag{1.46}$$

while with our realization of the a, a^+, N for q a root of unity, from (1.44), (1.45a) and (1.45b) we get the cyclic finite-dimensional representations of the centreless q-Virasoro algebra [27, 28]:

$$L_n=g^n\frac{h-h^{-1}}{\omega-\omega^{-1}} \qquad n=0,1,2,\ldots,\kappa-1. \tag{1.47}$$

As we shall see later the q-oscillator algebra in the form (1.5a)–(1.5c) can be realized as a part of the q-orthosymplectic superalgebra osp(2.1) and this superalgebra being a Hopf algebra induces a Hopf structure to the q-Virasoro algebra (1.44), (1.45a)–(1.45c) in both cases (q being a root of unity or not).

2. Two or more q-oscillators

In the standard quantum-mechanical case, $q=1$, a free system of $M=1,2,\ldots,$ bosonic oscillators is defined through the algebra

$$[a_i, a_j^+]=\delta_{ij} \qquad [a_i, a_j]=[a_i^+, a_j^+]=0 \tag{2.1a}$$

$$N_i=a_i^+ a_i \qquad i,j=1,2,\ldots,M. \tag{2.1b}$$

the Hamiltonian is

$$H=\sum_{i=1}^{M} h_i \qquad h_i=N_i+\tfrac{1}{2} \tag{2.2}$$

and the symmetry of the system is U(M) with generators

$$E_{ij}=a_i^+ a_i \tag{2.3a}$$

$$N_i=a_i^+ a_i \tag{2.3b}$$

When $q\neq 1$ the algebra (2.1a), (2.1b) becomes

$$a_i a_i^+ - q a_i^+ a_i = q^{-N_i} \qquad a_i^+ a_i = \frac{q^{N_i}-q^{-N_i}}{q-q^{-1}} \tag{2.4a}$$

$$[a_i, a_j^+]=[a_i, a_j]=[a_i^+, a_j^+]=0 \qquad i\neq j=1,\ldots,M. \tag{2.4b}$$

The first thing we notice is that the free system of q-oscillators defined by the Hamiltonian

$$H=\sum_{i=1}^{M} H_i \qquad H_i=\tfrac{1}{2}(a_i^+ a_i + a_i a_i^+) \qquad i=1,\ldots,M \tag{2.5}$$

breaks the symmetry U(M) (this is expected), but also this symmetry is not transformed into a $\mathrm{U}_q(M)$ symmetry (as one would hope). To see this we take for simplicity $M=2$. Then relations (2.4a) and (2.4b) imply

$$H_i=\tfrac{1}{2}([h_i+\tfrac{1}{2}]+[h_i-\tfrac{1}{2}]) \qquad i=1,2 \tag{2.6a}$$

$$h_i=N_i+\tfrac{1}{2} \tag{2.6b}$$

$$H=H_1+H_2. \tag{2.6c}$$

To make the algebra transparent we put $q=\mathrm{e}^{\gamma}$, and so

$$H=\frac{\sinh(\gamma(h_1+h_2)/2)\cosh(\gamma(h_1-h_2)/2)}{2\sinh(\gamma/2)}. \tag{2.7}$$

The generators of the $\mathrm{SU}_q(2)$, which can be constructed out of a_1, a_1^+, a_2, a_2^+ are [10]

$$J_+=a_i^+ a_2 \qquad J_-=a_2^+ a_1 \qquad J_3=\tfrac{1}{2}(h_1-h_2) \tag{2.8}$$

and they satisfy the algebra

$$[J_3, J_\pm]=\pm J_\pm \qquad [J_+, J_-]=[2J_3]. \tag{2.9}$$

The quadratic Casimir of this algebra [10] (chosen to agree with the known Casimir for $q=1$) is

$$C_2=J_+J_-+[J_3-\tfrac{1}{2}]^2-\tfrac{1}{4} \tag{2.10a}$$

$$=\frac{\sinh^2(\gamma/2)(h_1+h_2)}{\sinh^2(\gamma/2)}-\tfrac{1}{4}. \tag{2.10b}$$

We observe that, when $\gamma=0$ ($q=1$),

$$C_2=\tfrac{1}{4}(H^2-1) \tag{2.11}$$

and so we have SU(2) symmetry (all generators commute with H). But when $\gamma\neq 0$ the Hamiltonian (2.7) depends explicitly on the generator J_3. The representation space of $SU_q(2)$ constructed from two q-oscillators is [9, 10, 19]

$$|j,m\rangle=\frac{(a_1^+)^{j+m}}{\sqrt{[j+m]!}}\frac{(a_2^+)^{j-m}}{\sqrt{[j-m]!}}|0\rangle \tag{2.12}$$

$$j+m=n_1 \qquad j-m=n_2 \qquad n_1, n_2=0,1,2,\ldots. \tag{2.13}$$

where n_1, n_2 are the eigenvalues of the number operators N_1, N_2. On this space the spectrum of the free Hamiltonian (2.7) with broken $SU_q(2)$ symmetry is as follows:

$$H|j,m\rangle=E_{j,m}|j,m\rangle \tag{2.14a}$$

$$E_{j,m}=\frac{\sinh(\gamma/2)(j+\frac{1}{2})\cosh\gamma m}{\sinh(\gamma/2)} \qquad j=0,1,\ldots \quad m=-j,\ldots,j. \tag{2.14b}$$

Now we pose the question of how it is possible to combine two oscillators in such a way that the resulting system exhibits $SU_q(2)$ symmetry (and in general M oscillators with $SU_q(M)$ symmetry). From the expressions (2.10a), (2.10b) for the Casimir of $SU_q(2)$ we see that we have to find a combination of Hamiltonians,

$$H_i=\frac{\sinh\gamma h_i}{2\sinh(\gamma/2)} \qquad i=1,2 \tag{2.15}$$

such that the total Hamiltonian is a function only of h_1+h_2. There is only one such linear combination of the two Hamiltonians (up to a permutation of 1 with 2) which gives the right limit when $q\to 1$,

$$H=H_1q^{h_2}+H_2q^{-h_1} \tag{2.16a}$$

$$H=\frac{\sinh\gamma(h_1+h_2)}{2\sinh(\gamma/2)}. \tag{2.16b}$$

For more than two oscillators, say M oscillators, one has to repeat the above construction and find

$$H=\sum_{\kappa=1}^{M} q^{-h_1-h_2\ldots-h_{\kappa-1}}H_\kappa q^{h_{\kappa+1}+\ldots+h_M} \tag{2.17}$$

or

$$H=\frac{\sinh\gamma(h_1+\ldots+h_M)}{2\sinh(\gamma/2)} \tag{2.18}$$

and

$$h_i=N_i+\tfrac{1}{2} \qquad i=1,\ldots,M. \tag{2.19}$$

This Hamiltonian is a $U_q(M)$ symmetry since it commutes with all generators of $SU_q(M)$ in the q-bosonic oscillator realization [11, 12]

$$E_{ij}=a_i^+a_j \qquad i\neq j=1,\ldots,M \tag{2.20a}$$

$$d_i=h_i-h_{i+1} \qquad i=1,\ldots,M-1 \tag{2.20b}$$

$$d=\sum_{i=1}^{M}h_i-\frac{M}{2} \tag{2.20c}$$

and the spectrum on the basis

$$|n_1; \ldots, n_M\rangle = \prod_{i=1}^{M} \frac{(a_i^+)^{n_i}}{\sqrt{[n_i]!}} |0\rangle. \tag{2.21}$$

$$n_1, n_2, \ldots = 0, 1, 2, \ldots.$$

is given by

$$H|n_1, \ldots, n_M\rangle = E_{n_1\ldots n_M}|n_1, \ldots, n_M\rangle \tag{2.22a}$$

$$E_{n_1\ldots n_M} = \frac{\sinh \gamma(n_1 + \ldots + n_M + M/2)}{2\sinh(\gamma/2)}. \tag{2.22b}$$

We observe that the above q-oscillator system is interacting. For $\gamma \neq 0$, we have repulsion of the levels. When $\gamma = i\alpha$, imaginary, some levels feel attraction and others repulsions. In particular when $-\pi < \alpha < \pi$, the ground state energy decreases with the number of oscillators.

It is possible to construct Hamiltonians with any product of q-symmetries. For example, if H_1, H_2 are Hamiltonians with $U_q(M_1)$, $U_q(M_2)$ symmetry then it is obvious that $H = H_1 + H_2$ has symmetry $U_q(M_1) \times U_q(M_2)$ and not $U_q(M_1 + M_2)$.

Finally, here are some remarks on the degeneracy of the $U_q(M)$ symmetric Hamiltonians. For generic γ (not $\gamma = 2\pi i/\kappa$) the degeneracy for every level is determined by a fixed value of $n_1 + \ldots + n_M$, say n, and it is equal to $d(n)$, the number of partitions of n into positive integers. When $\gamma = 2\pi i/\kappa$ then the symmetry increases with the degeneracy $d_\kappa(n)$ equal to the number of partitions of n into positive integers mod κ.

Another remark concerns the representation of $SU_q(M)$ (see 2.20a) and (2.20b)) in the case $q = e^{2\pi i/\kappa}$. Using the expressions for the a_i, a_i^+s given by the matrices g, h (see 1.30)) we obtain explicit matrix realizations for this 'singular' case [29]. For example, for $SU_q(2)$ we have (1.27)

$$J_+ = g \otimes g^{-1} \frac{h - h^{-1}}{\omega - \omega^{-1}} \qquad \omega = q = e^{2\pi i/\kappa} \tag{2.23a}$$

$$J_- = g^{-1} \frac{h - h^{-1}}{\omega - \omega^{-1}} \otimes g \tag{2.23b}$$

$$J_3 = \tfrac{1}{2}(P \otimes 1 - 1 \otimes P). \tag{2.23c}$$

This representation is irreducible and we have

$$J_+^\kappa = J_-^\kappa = 0. \tag{2.24}$$

3. Hopf algebra for the q-oscillator

After exposing the general strategy for the construction of $SU_q(M)$-invariant Hamiltonians, we would like to go back to the two q-oscillator system (or two-dimensional q-oscillator) and discuss the origin of the 'co-multiplication type' of combination of the single-oscillator Hamiltonians.

It has been noticed recently [15, 16, 18], that there is a q-bosonic realization of the orthosymplectic q-superalgebra osp_q (2.1) [30–32], with the a, a^+ operators playing the role of the odd elements and a^2, a^{+2} the role of the even elements. A minimal subalgebra of the algebra osp (2.1) is formed by the elements a, a^+, $h=N+\frac{1}{2}$:

$$[h,a]=-a \qquad [h,a^+]=a^+ \qquad \{a,a^+\}=\frac{\sinh \gamma h}{2\sinh(\gamma/2)}. \tag{3.1}$$

The superalgebra osp_q (2.1) has a Hopf structure. The co-multiplication being

$$\Delta(a)=a\otimes q^{h/2}+q^{h/2}\otimes a \tag{3.2a}$$

$$\Delta(a^+)=a^+\otimes q^{h/2}+q^{-h/2}\otimes a^+ \tag{3.2b}$$

$$\Delta(h)=h\otimes I+I\otimes h \tag{3.2c}$$

the antipode and the co-unit given by

$$\gamma(a^+)=-q^{1/2}a^+ \qquad \gamma(a)=-q^{-1/2}a \qquad \gamma(h)=-h \tag{3.3a}$$

$$\varepsilon(1)=1 \qquad \varepsilon(h)=\varepsilon(a)=\varepsilon(a^+)=0. \tag{3.3b}$$

The algebra (3.1) has Δ, as co-multiplication with a graded tensor product

$$(a\otimes b)(c\otimes d)=(-1)^{\delta(b)\delta(c)}ac\otimes bd \tag{3.4}$$

with

$$\delta(a)=\delta(a^+)=1 \qquad \delta(h)=0. \tag{3.5}$$

Even in the case $q=1$ the graded tensor product structure is needed because of the anticommutators. In the tensor product space

$$|n_1,n_2\rangle=\frac{(a_1^+)^{n_1}}{\sqrt{[n_1]!}}\frac{(a_2^+)^{n_2}}{\sqrt{[n_2]!}}|0,0\rangle \qquad n_1,n_2=0,1,\ldots \tag{3.6}$$

one may construct the states

$$|n\rangle=\frac{\Delta(a^+)^n}{\sqrt{[n]!}}|0,0\rangle \qquad n=0,1,2,\ldots \tag{3.7}$$

on which the operators $\Delta(a)$, $\Delta(a^+)$, $\Delta(h)$ satisfy the q-oscillator algebra

$$\Delta(a)\Delta(a^+)-q\Delta(a^+)\Delta(a)=q^{-\Delta(h)+1/2}. \tag{3.8}$$

Since (3.1) 'defines', the Hamiltonian, H

$$H=\tfrac{1}{2}\{a,a^+\} \tag{3.9}$$

we see that the co-multiplication defines the correct 'tensoring' of the two-oscillator Hamiltonians:

$$\Delta(H)=\tfrac{1}{2}\{\Delta(a),\Delta(a^+)\}=H_1q^{h_2}+q^{-h_1}H_2. \tag{3.10}$$

In this framework there is still an $\text{SU}_q(2)$ symmetry (a_1, a_2^+ anticommute with a_2, a_2^+),

$$J_+=a_1^+a_2 \qquad J_-=-a_2^+a_1 \qquad J_3=\tfrac{1}{2}(h_1-h_2) \tag{3.11}$$

which commutes with $\Delta(H)$.

One may, however, forget this superalgebra structure as is done in the $q=1$ case when the two bosonic oscillator system is discussed. Since (3.10) gives the $\text{SU}_q(2)$ symmetric Hamiltonian, even when the a_1, a_1^+ commute with the a_2, a_2^+, we may consider the Hopf algebraic structure only as a helpful device, although we think that it would be interesting to examine systems of oscillators, which have anticommuting a_i, a_i^+ operators.

4. Conclusion

We have defined many-body systems of q-oscillators with q-symmetries. Although these systems seem to be non-local, there are interesting physical systems with many-body non-local interactions. Such systems appear in $2+1$ dimensions in condensed matter physics—the now famous anyons [33, 34]. It is known that the anyon system exhibits exotic statistics with braid group permutation symmetry. Because of this and because the braid group plays a distinguished role in the representations of $\mathrm{SU}_q(M)$ groups, which are the symmetries of the presented q-oscillator systems, we believe that the latter must possess, apart from their exact integrability, analogous physical properties with the anyon systems. In some recent work [23] we found a relation of the one q-oscillator system and the angular motion of the two-anyon system. It could be the case that more degrees of freedom of the anyon systems can be implemented algebraically using the many q-oscillator systems with $\mathrm{SU}_q(M)$ symmetries.

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